Lecture 3: abelian symmetries in bSM models

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Outline

1. Abelian groups: introduction
2. Finding abelian symmetries
3. An example: 3HDM with quarks
4. Listing possible symmetries

Building bSM models

- Abelian groups are the simplest type of groups; they are easier to study than non-abelian.
- Abelian groups are building blocks of all groups; studying them within bSM models, we learn something about models with non-abelian groups as well.
- Abelian groups, continuous or discrete, are very often used in bSM phenomenology.

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- Abelian groups, continuous or discrete, are very often used in bSM phenomenology.

It makes perfect sense to study in detail
how abelian symmetries shape the bSM models.

Basics of abelian groups

Now, brace for some (simple) maths.

Basics of abelian groups

Group A is abelian if all its elements commute: $ab = ba$ for any $a, b \in A$.

Example:

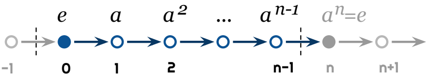
- Integers $\mathbb{Z}$ and reals $\mathbb{R}$ under addition, as well as $\mathbb{R}\backslash\{0\}$ under multiplication.
- Reals on the interval $[0, 1]$ under addition and with periodic boundary condition (= fractional part of reals): $\mathbb{R}/\mathbb{Z}$. Circle group (complex numbers with $|z| = 1$ under multiplication) = rephasing group $U(1)$ $\simeq \mathbb{R}/\mathbb{Z}$.

Basics of abelian groups

- Cyclic groups $\mathbb{Z}_n$ for any $n > 1$, which are defined as

$$\mathbb{Z}_n = \{e,\, a,\, a^2,\, a^3,\, \ldots,\, a^{n-1}\} \quad \text{with condition} \quad a^n = e\,,$$

which are isomorphic to integers modulo n under addition: $\mathbb{Z}/n\mathbb{Z}$. The entire group is generated by a: $\mathbb{Z}_n = \langle a \,|\, a^n = e \rangle$.



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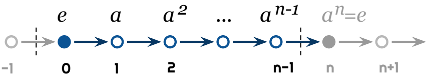
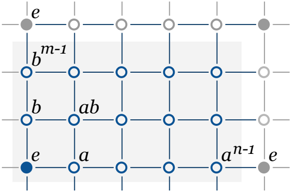


- Direct products of cyclic groups, such as

$$\mathbb{Z}_n \times \mathbb{Z}_m = \langle a,\, b\,|\,a^n = b^m = e,\, ab = ba\rangle.$$



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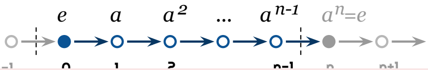

All finite abelian groups are direct products of cyclic groups; infinite abelian groups are direct products of cyclic group, $\mathbb{Z}$'s, $\mathbb{R}$'s, and/or $U(1)$'s.

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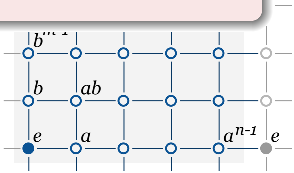

Basics of abelian groups

Be careful: despite groups $\mathbb{Z}_{nm}$ and $\mathbb{Z}_n \times \mathbb{Z}_m$ are two abelian groups with the same order $|\mathbb{Z}_{nm}| = |\mathbb{Z}_n \times \mathbb{Z}_m| = nm$, they are, generally speaking, different.

More precisely, $\mathbb{Z}_{nm} \simeq \mathbb{Z}_n \times \mathbb{Z}_m$ if and only if the greatest common divisor of n and m is 1. For example,

$$\mathbb{Z}_2 \times \mathbb{Z}_3 \simeq \mathbb{Z}_6\,, \quad \mathbb{Z}_4 \times \mathbb{Z}_{25} \simeq \mathbb{Z}_{100}\,.$$

while groups $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_8$ are all different.

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Indeed, take $\mathbb{Z}_2 \times \mathbb{Z}_3 = \langle a,\, b \,|\, a^2 = b^3 = e,\, ab = ba\rangle$ and consider $c = ab$:

$$c^4 = a^4 b^4 = b \,, \quad c^3 = a^3 b^3 = a \,.$$

The entire group can be written via powers of generator c.

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Questions

Q3.1: try to apply this argument to $\mathbb{Z}_2 \times \mathbb{Z}_4$ and see why it fails.
Q3.2: Let G be a group whose *every* element $g \in G$ satisfies $g^2 = e$. Show that G is abelian.

Basics of abelian group representations

Consider N-dimensional complex vector space $\mathbb{C}^N$ of vectors $x = (x_1, \ldots, x_N)$. The group of unitary transformations of this space is $U(N)$. If we can find a copy of any group G inside $U(N)$, then we say that vectors x realize an N-dimensional unitary representation of the group G.

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$U(N)$ is non-abelian, but it contains many abelian subgroups: A_1, A_2, etc. Some of these abelian subgroups can lie in larger abelian subgroups: $A_1 \subset A_2 \subset \cdots \subset U(N)$. Abelian subgroups of $U(N)$ which do not are called maximal abelian subgroups.

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NB: "maximal" in the sense of containment, not in the sense of size!

Basics of abelian group representations

All maximal abelian subgroups of $U(N)$ are $U(1)\times\cdots\times U(1) = [U(1)]^N$ and are isomorphic to the rephasing group $U(1)_1\times\cdots\times U(1)_N$, where

$$\begin{aligned}
U(1)_1: &\quad \mathrm{diag}(e^{i\alpha_1},\,1,\,\dots,\,1,\,1)\,,\\
U(1)_2: &\quad \mathrm{diag}(1,\,e^{i\alpha_2},\,\dots,\,1,\,1)\,,\\
&\qquad\vdots\\
U(1)_N: &\quad \mathrm{diag}(1,\,1,\,\dots,\,1,\,e^{i\alpha_N})\,,\quad \alpha_i = [0,\,2\pi]\,.
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The abelian groups we usually work with are built of cyclic groups and $U(1)$'s and admit a unitary representation of appropriate dimension N.

For example, $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset U(1)_1 \times U(1)_2$ can be represented by matrices $\mathrm{diag}(-1, 1)$ and $\mathrm{diag}(1, -1)$ and their products.

Basics of abelian group representations

Suppose we have an N-dim. representation of the group G. If there exists a subspace of $\mathbb{C}^N$ invariant under the entire group G, then this representation is reducible; if not, then it is irreducible (irrep).

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For example, if $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b | a^2 = b^2 = e \rangle$, then we can define the following 1D irrep: $a \mapsto -1$, $b \mapsto 1$. Is it OK for our purposes of bSM model building?

Basics of abelian group representations

Not really. Beyond rep/irrep, there is another important property of a representation: being (un)faithful. The representation is called faithful, if different elements of the group G correspond to different matrices.

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In the example, we have

$$A = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b \,|\, a^2 = b^2 = e \rangle \,, \quad a \mapsto -1, \;\; b \mapsto 1 \,,$$

two distinct elements e and b are mapped to the same representing element of $U(1)$. Although it is an irrep of A, it is unfaithful; it does not really implement the full group A inside $U(1)$.

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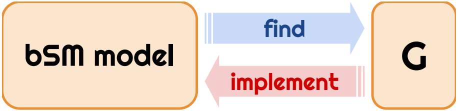
When constructing and analyzing models with some symmetry groups, it makes sense first to check whether a representation is faithful, and only then distinguish between reps and irreps.

Finding vs. implementing

When we talk about symmetries in bSM models, we can study two opposite questions:

- Given a particular bSM model, find its symmetry group G,
- Given a symmetry group G, implement this group into some bSM model.



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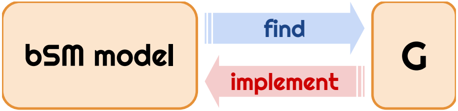

Usually, model-builders focus on implementing because this step is easy (and often guided by pure guess) and because they want to proceed to phenomenology as quickly as possible.

Finding vs. implementing

The task of finding G is also important:

- it gives a check that the constructed model indeed implements the desired symmetry group and nothing bad on top of it;
- it shows the limits of implementation of G in models: what can be implemented in a given class of models, and what cannot.

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Suppose you implement a group G. You introduces fields and write $\mathcal{L}$ invariant under G. However, it might happen that the resulting $\mathcal{L}$ is automatically invariant under a larger group $G' \supset G$, which you didn't intend to implement. This extra symmetry can be good or bad:

- good, when this "extra symmetry for free" explains some patterns which you didn't know how to explain within G;
- bad, when it leads to undesirable consequences e.g. massless scalars.

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In lectures 3 and 4 I will focus on the task of finding symmetry group of a model:

- Lecture 3: general procedure for abelian groups and arbitrary models,
- Lecture 4: some examples in 3HDM with non-abelian symmetry groups.

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General procedure

There exists a powerful method for finding the rephasing symmetry group of any interaction lagrangian.

It uses some simple matrix algebra which is, surprisingly, not very well known in bSM community, namely, the Smith normal form of an integer-valued matrix, which transparently encodes all rephasing symmetries of any lagrangian.

I will illustrate it with rephasing symmetries of the scalar sector multi-Higgs-doublet models, but keep in mind that this is a universal method.

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First mentioned in *[Petersen, Ratz, Schieren, 2009]* (remnant discrete symmetries in GUT models); developed much further in: *[Ivanov, Keus, Vdovin, 2012]* for NHDM scalar sector, and *[Ivanov, Nishi, 2013]* for NHDM with quarks.

Rephasing symmetries in NHDM

With N doublets, the total rephasing group is $[U(1)]^N$, with each $U(1)$ generated by individual doublet rephasing $\phi_j \mapsto e^{i\alpha_j}\phi_j$.

Suppose that the potential $V = Y_{ab}(\phi_a^\dagger\phi_b) + Z_{abcd}(\phi_a^\dagger\phi_b)(\phi_c^\dagger\phi_d)$ has k "rephasing-sensitive" terms.

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$$(\phi_1^\dagger\phi_2)(\phi_1^\dagger\phi_3) \quad \mapsto \quad e^{i(-2\alpha_1+\alpha_2+\alpha_3)}(\phi_1^\dagger\phi_2)(\phi_1^\dagger\phi_3)\,.$$

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Write this phase as $\sum_{j=1}^N d_{1j}\alpha_j$, where $d_{1j} = (-2,\,1,\,1,\,0,\,\dots,\,0)$. Then angles α_j satisfying

$$d_{1j}\alpha_j = 2\pi n_1$$

with any integer n_1 leave this term invariant.

Rephasing symmetries in NHDM

Repeat it for every other term in V; the i-th term produces d_{ij}.

Rephasing symmetries of the potential are generated by solutions α_j of the following system of equations:

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A 4HDM example:

$$V = V_{\text{reph.inv.}} + \lambda_1(\phi_4^\dagger\phi_1)(\phi_3^\dagger\phi_1) + \lambda_2(\phi_4^\dagger\phi_2)(\phi_1^\dagger\phi_2) + \lambda_3(\phi_4^\dagger\phi_3)(\phi_2^\dagger\phi_3) + \text{h.c.}$$

gives

$$d_{ij} = \begin{pmatrix} 2 & 0 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ 0 & -1 & 2 & -1 \end{pmatrix}.$$

Rephasing symmetries in NHDM

The matrix d_{ij} always has integer entries. By certain elementary steps

- permutation of rows or columns,
- sign flips of rows or columns,
- adding a column/row to another column/row

it can be diagonalized:

$$d_{ij} \quad \mapsto \quad \begin{pmatrix} d_1 & & & & & & \\ & d_2 & & & & & \\ & & \ddots & & & & \\ & & & d_r & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 & \cdots \end{pmatrix}, \qquad r = \mathrm{rank}\, d_{ij}$$

with all d_i positive and such that d_i divides d_{i+1}.

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with all d_i positive and such that d_i divides d_{i+1}.

This is known as the Smith Normal Form (SNF) of d_{ij}.
It exists and is unique for any integer-valued matrix.

Rephasing symmetries in NHDM

The key observation: elementary steps do not change the set of solutions.

The equations become decoupled; each $d_i\tilde{\alpha}_i = 2\pi\tilde{n}_i$ has solutions $\tilde{\alpha}_i = 2\pi\tilde{n}_i/d_i$, which generate the group $\mathbb{Z}_{d_i}$.

The rephasing group of the potential is therefore

$$A = \mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2} \times \cdots \times \mathbb{Z}_{d_r} \times [U(1)]^{N-r}\,.$$

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Let's apply it to the 4HDM example

$$V = V_{\text{reph.inv.}} + \lambda_1(\phi_4^\dagger\phi_1)(\phi_3^\dagger\phi_1) + \lambda_2(\phi_4^\dagger\phi_2)(\phi_1^\dagger\phi_2) + \lambda_3(\phi_4^\dagger\phi_3)(\phi_2^\dagger\phi_3) + \mathrm{h.c.}$$

gives

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4HDM example

$$\begin{pmatrix} 2 & 0 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ 0 & -1 & 2 & -1 \end{pmatrix} \to \begin{pmatrix} 2 & 0 & -1 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & -1 & 0 \\ -1 & 2 & 0 & 0 \\ 4 & -1 & 2 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 4 & 7 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 7 & 0 \end{pmatrix}.$$

The rephasing symmetry group is $A = \mathbb{Z}_7 \times U(1)$, where $\mathbb{Z}_7$ is generated by

$$\alpha_i = \frac{2\pi}{7}(0,\, 1,\, -2,\, 2)\,,$$

while $U(1)$ is just the overall rephasing with $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$.

3HDM with quarks

Consider the following 3HDM with a rephasing-invariant Higgs potential and with the following Yukawa interactions with quarks:

$$-\mathcal{L}_Y = \Gamma^{(j_\phi)}_{j_L j_d} \bar{Q}_{L j_L} \phi_{j_\phi} d_{R j_d} + \Delta^{(j_\phi)}_{j_L j_u} \bar{Q}_{L j_L} \tilde{\phi}_{j_\phi} u_{R j_u} + h.c.$$

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We order the 12 fields as $(\phi_{j_\phi};\ Q_{L j_L};\ d_{R j_d};\ u_{R j_u})$, where $j_\phi,\ j_L,\ j_d,\ j_u = 1, 2, 3$, and we choose the following textures:

$$\Gamma^{(1)} = \begin{pmatrix} 0 & 0 & * \\ 0 & * & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Gamma^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & 0 & 0 \end{pmatrix}, \quad \Gamma^{(3)} = \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix},$$

$$\Delta^{(1)} = \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & * \end{pmatrix}, \quad \Delta^{(2)} = \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & 0 \\ * & 0 & 0 \end{pmatrix}, \quad \Delta^{(3)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix}.$$

There are 12 Yukawa terms; 6 with d_R's and 6 with u_R's.

3HDM with quarks

Each Yukawa term produces a row d_{ij} with entries ± 1 or 0.

For example, the term with $\Gamma^{(1)}_{13}$ is $\bar{Q}_{L1}\phi_1 d_{R3}$, and its row d_{ij} is

$$\big(\overbrace{1,\ 0,\ 0}^{\phi}\ \big|\ \overbrace{-1,\ 0,\ 0}^{Q_L}\ \big|\ \overbrace{0,\ 0,\ 1}^{d_R}\ \big|\ \overbrace{0,\ 0,\ 0}^{u_R}\big)\,,$$

and the term with $\Delta^{(2)}_{31}$ is $\bar{Q}_{L3}\tilde{\phi}_2 u_{R1}$, and its row d_{ij} is

$$\big(\,0\ -1,\ 0\ |\ 0,\ 0,\ -1\ |\ 0,\ 0,\ 0\ |\ 1,\ 0,\ 0\,\big)\,.$$

3HDM with quarks

The entire matrix d_{ij} is a 12×12 matrix:

$$d_{ij} = \left(\begin{array}{ccc|ccc|ccc|ccc}
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right).$$

3HDM with quarks

Its Smith normal form is

$$\mathrm{diag}(1, 1, 1, 1, 1, 1, 1, 1, 1, 5, 0, 0)\,.$$

Question

Q3.3: prove it!

3HDM with quarks

Its Smith normal form is

$$\mathrm{diag}(1, 1, 1, 1, 1, 1, 1, 1, 1, 5, 0, 0)\,.$$

Question

Q3.3: prove it!

The symmetry group is

$$G = \mathbb{Z}_5 \times U(1)_Y \times U(1)_B\,.$$

The $\mathbb{Z}_5$ generator is given by rephasing:

$$\alpha_i = \frac{2\pi}{5}(0, 2, 4\,|\,2, 1, 0\,|\,3, 1, 2\,|\,2, 4, 0)\,.$$

Beyond case-by-case checks

In principle, the SNF technique gives the rephasing symmetry group for any interaction lagrangian. Suppose we want to know all possible abelian symmetry groups for a given field content.

Naively, we would need to write down all possible interaction terms, then consider all possible combinations of these terms, for each combination construct d_{ij}, and then find its SNF. Is it feasible?

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Example: scalar sector of 3HDM: 3 doublets, 36 rephasing sensitive terms, 2^{36} possible combinations of these interactions terms! But do we really need to check all these zillions of combinations?

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> No!
> The matrix d_{ij} itself offers the answer.

Beyond case-by-case checks

The point is that d_{ij} typically has rows of very simple structure.

Scalar sector in NHDM:

$$(2,\,-2,\,0,\,0,\,\dots)\,,\quad (2,\,-1,\,-1,\,0,\,\dots)\,,\quad (1,\,1,\,-1,\,-1,\,0,\,\dots)\,,$$

up to permutations.

Beyond case-by-case checks

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Quarks Yukawa interactions with Higgs doublets:

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Usually, one can find all possible values of $|\det d|$.

Beyond case-by-case checks

$|\det d|$ remains unchanged after we bring it to the SNF:

$$|\det d| = \det D_{\rm SNF} = \prod_j d_j\,.$$

Indeed, $D_{\rm SNF} = R \cdot d \cdot C$, where R and C encode manipulations with rows and columns, respectively. For example, $C = C_1 C_2 \cdots C_p$, product of elementary manipulations with columns, which can be

$$C_i = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & 1 & \\ & & & 1 & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & 1 & \\ & & 1 & 0 & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & -1 & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

In any case, $|\det C_i| = 1$, so $|\det C| = |\det R| = 1$.

Beyond case-by-case checks

We then get a handy procedure of guessing the symmetry group without even calculating SNF:

- get rid of all "automatic" $U(1)$'s (to avoid $d_i = 0$);
- using the structure of d_{ij}, find all values of $|\det d| = |A|$;
- if the resulting $|A|$ is a number whose prime decomposition involves only first powers of primes, then the group A is uniquely determinted;
- if the prime decomposition of $|A|$ involves higher powers, then ambiguity exists; only in this case one needs to explicitly find SNF.

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For example,

- if $|A| = 5$, then the group A must be $\mathbb{Z}_5$;
- if $|A| = 30$, then the group A must be $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$;
- if $|A| = 4$, then the group A can be either $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$. One needs to check whether SNF is $(\ldots, 1, 2, 2)$ or $(\ldots, 1, 1, 4)$.

Beyond case-by-case checks

Additionally, one can also obtain the general upper bound on $|A|$.

- scalar sector of NHDM: $|A_\phi| \leq 2^{N-1}$ for any N;
- NHDM with quarks: $|A_{\phi,q}| \leq (N+1)^2/3$ for any N.

So, if we want to avoid continuous symmetries, then we have only the following abelian groups available:

- 2HDM: scalar sector with $\mathbb{Z}_2$ only; scalar + quark sector with $\mathbb{Z}_2$ or $\mathbb{Z}_3$, *[Ferreira, Silva, 2011]*;
- 3HDM: scalar sector with any group up to $\mathbb{Z}_4$; scalar + quark sector with any group up to $\mathbb{Z}_5$.

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Trying to implement any larger discrete group will unavoidably produce a model with continuous symmetry.

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  v …

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mode …

Much of this analysis can be done manually, without computer-algebra assistance.

What seemed to be a hugely complicated computer-assisted calculation, turns into an easy arithmetical exercise!

NNI texture for quark masses

Consider the next-neighbour interaction (NNI) texture for quark masses:

$$M_d \sim M_u \sim \begin{pmatrix} 0 & * & 0 \\ * & 0 & * \\ 0 & * & * \end{pmatrix} .$$

It was shown in *[Branco, Emmanuel-Costa, Simoes, 2010]* that if one wants to produce NNI texture within multi-doublet models purely from symmetries, the minimal realization is 2HDM with $\mathbb{Z}_4$, with the following Yukawa matrices:

$$\Gamma^{(1)} = \Delta^{(2)} = \begin{pmatrix} 0 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & * \end{pmatrix} , \quad \Gamma^{(2)} = \Delta^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix} .$$

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The above analysis shows that it is misleading to identify $\mathbb{Z}_4$ with a relevant symmetry group of this model. The model must have an extra continuous symmetry group, and $\mathbb{Z}_4$ is just one of its many subgroups.

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Question

Q3.4: prove that, in addition to $U(1)_Y$ and $U(1)_B$, this model has an extra $U(1)$ global symmetry. Find the corresonding "charges" q of all fields under this extra $U(1)$.

It was ... produ... e minima...

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Lecture 3 Summary

- Apart from constructing a bSM model with a given symmetry group G, one should also attack the reverse problem: finding the symmetry group of a given bSM model. This task is usually much more complicated, but any progress in this direction is rewarding.
- The rephasing symmetry group of any model can be determined via the Smith normal form technique, a simple, powerful, but not very well known method.
- Application of this technique to NHDM, both in the Higgs sector only or with Yukawa sectors, leads to results which are extremely difficult to obtain via traditional methods.