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F. Landis Markley

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[1] H. Goldstein, *Classical Mechanics* (Addison–Wesley, Reading, Mass., 1950); K. Symon, *Mechanics* (Addison–Wesley, Reading, Mass., 1971).

[2] Equation (4) results by subtraction of the time derivative of Eq. (2) from the time derivative of Eq. (3). Alternatively, Euler's equations could have been used to obtain Eqs. (4) and (5) directly.

Probability Distribution of Momenta in an Infinite Square-Well Potential

F. LANDIS MARKLEY
Thompson Physical Laboratory
Williams College
Williamstown, Mass 01267
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One of the first problems solved in any quantum mechanics course is the infinite square-well potential,

$$V(x)=0, \qquad |x|\le a,$$
$$=\infty, \qquad |x|>a.$$

The normalized energy eigenfunctions are well known to be

$$\psi_n(x)=0, \qquad |x|>a,$$
$$=a^{-1/2}\cos k_nx, \qquad |x|\le a,\ n \text{ odd},$$
$$=a^{-1/2}\sin k_nx, \qquad |x|\le a,\ n \text{ even},$$

where

$$k_n=n\pi/2a, \qquad n=1,2,3,\cdots.$$

If we use the complex exponential representation for the sine and cosine, we have for $|x|\le a$

$$\psi_n(x)=\tfrac{1}{2}a^{-1/2}[\exp(ik_nx)+\exp(-ik_nx)], \qquad n \text{ odd},$$
$$=(2i)^{-1}a^{-1/2}[\exp(ik_nx)-\exp(-ik_nx)], \qquad n \text{ even}.$$

At this point, one is tempted to say that "if we measure the momentum of a particle in a box, we expect to find either $p=+\hbar k_n$ or $p=-\hbar k_n$, and the two possibilities are equally likely."[1] This statement errs in ignoring the fact that the wave functions are only sinusoidal inside the potential well and are identically zero outside. A sum of two plane waves must give a periodic function, not one contained in a finite region of coordinate space.

In fact, the probability of finding the particle in the nth energy eigenstate with momentum between p and $p+dp$ is given by $P_n(p)dp$ where

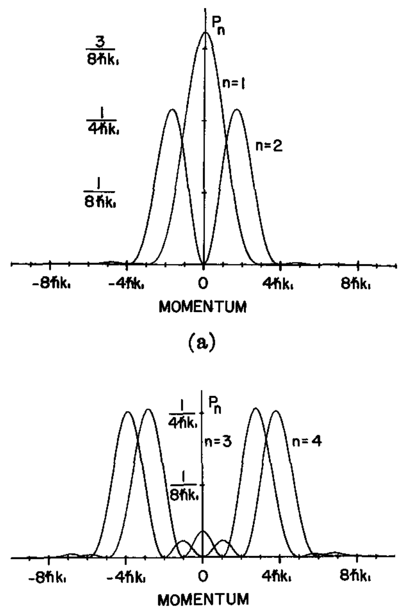

FIG. 1. Probability functions $P_n(p)$ vs momentum for the lowest four energy eigenstates of the infinite square-well potential. $k_1=\pi/2a$. (a) $n=1$ and $n=2$. (b) $n=3$ and $n=4$.

$$P_n(p)=|\phi_n(p)|^2.$$

Here $\phi_n(p)$ is the momentum space representation of the nth energy eigenstate and is given by the Fourier transform of the coordinate space wave function,

$$\phi_n(p)=(2\pi\hbar)^{-1/2}\int_{-\infty}^{\infty}\psi_n(x)\exp(ipx/\hbar)dx.$$

For the infinite square-well we have

$$\phi_n(p)=ni^{n-1}(\pi a/2\hbar)^{1/2}\cos(pa/\hbar)\times[(n\pi/2)^2-(pa/\hbar)^2]^{-1}, \qquad n \text{ odd},$$
$$=ni^{n-1}(\pi a/2\hbar)^{1/2}\sin(pa/\hbar)\times[(n\pi/2)^2-(pa/\hbar)^2]^{-1}, \qquad n \text{ even},$$

and

$$P_n(p) = (4\pi a n^2/\hbar)[1-(-1)^n \cos(2pa/\hbar)] \times[(n\pi)^2-(2pa/\hbar)^2]^{-2}.$$

The probability distributions $P_n(p)$ for $n=1, 2, 3, 4$ are plotted in Fig. 1. We see that most of the probability is concentrated in two asymmetric peaks of width $2\pi\hbar/a$ centered at the naively expected values $\pm\hbar k_n$ (the two peaks overlap for $n=1$), so the statement quoted above contains a grain of truth.

Finally, note that the calculation of the expectation value of the kinetic energy, which is almost trivial in the coordinate representation, becomes a rather formidable task in the momentum representation.

[1] This incorrect statement is found in Paul A. Tipler, *Foundations of Modern Physics* (Worth, New York, 1969), p. 245. The error is undoubtedly not confined to this book, which is generally quite clear and accurate in its presentation of quantum ideas.

Erratum: "Phase Waves of Louis deBroglie," JARED W. HASLETT, translator. [Amer. J. Phys. **40**, 1315 (1972)]. The institutional affiliation of Jared W. Haslett was incorrectly given in the explanatory note on p. 1315. Jared W. Haslett is affiliated with the University of Illinois at Chicago Circle, Department of Physics, Chicago, Illinois 60680.